

ABELIAN GRADINGS IN LIE ALGEBRAS

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ABSTRACT. Given a Lie algebra L graded by a group G , if L does not contain orthogonal graded ideals and G is generated by the support of L , then G is an abelian group.

It is well-known that the subgroup generated by the support of a simple G -graded Lie algebra is always abelian. Nevertheless, this fact can be easily extended to Lie algebras without orthogonal graded ideals, and this is the aim of this short note.

Throughout this note we deal with Lie algebras L over arbitrary rings of scalars Φ with $\frac{1}{2} \in \Phi$, with Lie bracket denoted by $[\cdot, \cdot]$. Let G be a group. We say that a Lie algebra L is graded by G if there exists a decomposition

$$L = \bigoplus_{g \in G} L_g$$

where each L_g is a Φ -submodule of L satisfying $[L_g, L_{g'}] \subset L_{gg'}$, for every $g, g' \in G$. The support of an element $a = \sum_{g \in G} a_g \in L$ is the finite set $\text{supp}(a) = \{g \in G \mid a_g \neq 0\}$, and the support of L as a G -graded algebra is the set $\text{supp}(L) = \bigcup_{a \in L} \text{supp}(a)$.

The next proposition relates noncommutative elements of G with orthogonal ideals of a G -graded Lie algebra L . We say that a G -graded Lie algebra L is graded-prime if it does not contain graded ideals $I, J \triangleleft L$ such that $[I, J] = 0$.

Proposition *Let L be a Lie algebra graded by a group G and let $g, g' \in G$ such that $gg' \neq g'g$. Then $[\text{id}_L(L_g), \text{id}_L(L_{g'})] = 0$. In particular, if L is graded-prime and G is generated by the support of L , G is an abelian group.*

Proof. Let us prove the following property: Let $g_1, g_2, \dots, g_n \in G$

$$\text{if } [L_{g_1}, [L_{g_2}, [\dots, [L_{g_{n-1}}, L_{g_n}]]]] \neq 0 \quad \text{then } g_i g_j = g_j g_i \quad \forall i, j \in \{1, 2, \dots, n\}. \quad (*)$$

This property is true for $n = 2$:

$$L_{g_1 g_2} \supset [L_{g_1}, L_{g_2}] = [L_{g_2}, L_{g_1}] \subset L_{g_2 g_1}$$

which implies, if $0 \neq [L_{g_1}, L_{g_2}] \subset L_{g_1 g_2} \cap L_{g_2 g_1}$, that $g_1 g_2 = g_2 g_1$. Let us suppose that $(*)$ is true for every $n - 1$ elements and let $0 \neq [L_{g_1}, [L_{g_2}, [\dots, [L_{g_{n-1}}, L_{g_n}]]]]$. By hypothesis,

$$g_i g_j = g_j g_i \quad \text{if } i, j \in \{2, 3, \dots, n\} \quad (1)$$

By the Jacobi identity we have two possibilities:

2000 *Mathematics Subject Classification.* Primary 17B05; Secondary 17B60.

The first author was partially supported by the MEC and Fondos FEDER MTM2007-62390 and MTM2010-16153, by FMQ 264, and by MICINN-I3-2010/00075/001.

The second author was partially supported by the MEC and Fondos FEDER MTM2007-61978 MTM2010-19482, by FMQ 264 and FQM 3737, and by MICINN-I3-2010/00075/001.

a). If $0 \neq [[L_{g_1}, L_{g_2}], [\dots, [L_{g_{n-1}}, L_{g_n}]]] \subset [L_{g_1 g_2}, [\dots, [L_{g_{n-1}}, L_{g_n}]]]$ then $g_1 g_2 = g_2 g_1$ and, by the induction hypothesis and (1), the elements $g_1 g_2$ and g_2 commute with every g_k , $k \geq 3$. Then $g_k(g_1 g_2) = (g_1 g_2)g_k = g_1 g_k g_2$ and multiplying by g_2^{-1} on the right we get $g_1 g_k = g_k g_1$ for every $k \geq 3$.

b). If $[L_{g_2}, [L_{g_1}, [\dots, [L_{g_{n-1}}, L_{g_n}]]]] \neq 0$, take $h = g_3 \dots g_n$. By induction,

$$g_1 h = h g_1, \quad g_2(g_1 h) = (g_1 h)g_2, \quad \text{and } g_1 g_k = g_k \text{ for every } k \geq 3,$$

so by (1), $g_2(g_1 h) = (g_1 h)g_2 = g_1 g_2 h$, which implies $g_1 g_2 = g_2 g_1$.

Now, if $g, g' \in G$ with $gg' \neq g'g$, then for every $g_1, g_2, \dots, g_n \in G$

$$[L_{g'}, [L_{g_1}, [\dots, [L_{g_n}, L_g]]]] = 0$$

which proves that $[\text{id}_L(L_{g'}), \text{id}_L(L_g)] = 0$. □

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